Some Further Results on Hermitian-Matrix Inequalities

Jerzy K. Baksalary
Department of Mathematics
Tadeusz Kotarbiński Pedagogical University
PL–65-625 Zielona Góra, Poland
and
Department of Mathematical Sciences
University of Tampere
P.O. Box 607
SF-33101 Tampere, Finland

and

Bernhard Schipp and Götz Trenkler
Department of Statistics
University of Dortmund
D-4600 Dortmund, Federal Republic of Germany

Submitted by George P. H. Styan

ABSTRACT

Some new results on the Löwner partial ordering between certain sums, products, and direct products of matrices are derived. Proofs of these results are based on a necessary and sufficient condition for the nonnegative definiteness of the difference of two Hermitian matrices, which follows from the criterion for the nonnegative definiteness of a partitioned Hermitian matrix due to Albert (1969).

1. INTRODUCTION

Let $\mathbb{C}_{m,n}$ denote the set of $m \times n$ complex matrices, let $\mathbb{H}_m$ denote the set of $m \times m$ Hermitian matrices, and let $\mathbb{V}_m$ denote the subset of $\mathbb{H}_m$ consisting of nonnegative definite matrices, i.e., $\mathbf{A} \in \mathbb{V}_m$ if $\mathbf{A} = \mathbf{A}^*$, where $\mathbf{A}^*$ is the conjugate transpose of $\mathbf{A}$, and $\mathbf{x}^*\mathbf{A}\mathbf{x} \geqslant 0$ for all $\mathbf{x} \in \mathbb{C}_{m,1}$. For $\mathbf{A}, \mathbf{B} \in \mathbb{H}_m$

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we will write $\mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B}$ if $\mathbf{B}-\mathbf{A} \in \mathbb{V}_m$, i.e., if $\mathbf{A}$ is below $\mathbf{B}$ with respect to the Löwner partial ordering.

Our purpose is to derive some new results on the Löwner ordering. In Section 3, which is the main part of the paper, we generalize (in our Theorem 2) the results of Man (1970), concerned with relationships between $\mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B}$ and $\mathbf{ACA} \leqslant_{\mathrm{L}} \mathbf{BCB}$, where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{V}_m$, and (in Theorem 3) the results of Taylor (1976) and Baksalary, Kala, and Kłaczyński (1983), concerned with the ordering $\mathbf{A}^*\mathbf{BA} \leqslant_{\mathrm{L}} \mathbf{B}$, where $\mathbf{A} \in \mathbb{C}_{m,m}$ and $\mathbf{B} \in \mathbb{V}_m$. Furthermore, we derive (in our Theorems 4 and 5, respectively) characterizations of the relation $\mathbf{0} \leqslant_{\mathrm{L}} \mathbf{AB}^* + \mathbf{BA}^*$, where $\mathbf{A}$, $\mathbf{B} \in \mathbb{C}_{m,n}$, and the relation $\mathbf{A} \otimes \mathbf{C} \leqslant_{\mathrm{L}} \mathbf{B} \otimes \mathbf{D}$, where $\mathbf{A}, \mathbf{B} \in \mathbb{V}_m$, $\mathbf{C}, \mathbf{D} \in \mathbb{V}_n$, and $\otimes$ denotes the direct (Kronecker) product. Our derivations are based on a necessary and sufficient condition for $\mathbf{A}_1\mathbf{A}_1^* \leqslant_{\mathrm{L}} \mathbf{B}$, where $\mathbf{A}_1 \in \mathbb{C}_{m,n}$ and $\mathbf{B} \in \mathbb{H}_m$, which is given in Theorem 1 as a simple consequence of the criterion for the nonnegative definiteness of a partitioned Hermitian matrix due to Albert (1969). Relationships of this condition to other results known in the literature are discussed in Section 2.

2. BASIC RESULT

If a Hermitian matrix $\mathbf{H}$ is partitioned as

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{12}^* & \mathbf{H}_{22} \end{pmatrix},$$

then

$$\mathbf{0} \leqslant_{\mathrm{L}} \mathbf{H} \quad \Leftrightarrow \quad \mathbf{0} \leqslant_{\mathrm{L}} \mathbf{H}_{11}, \quad \mathscr{R}(\mathbf{H}_{12}) \subseteq \mathscr{R}(\mathbf{H}_{11}), \text{ and } \mathbf{H}_{12}^*\mathbf{H}_{11}^-\mathbf{H}_{12} \leqslant_{\mathrm{L}} \mathbf{H}_{22}, \tag{1}$$

where $\mathscr{R}(\cdot)$ denotes the range and $\mathbf{H}_{11}^-$ is any generalized inverse of $\mathbf{H}_{11}$, i.e., $\mathbf{H}_{11}\mathbf{H}_{11}^-\mathbf{H}_{11} = \mathbf{H}_{11}$. The equivalence (1) was originally proved by Albert (1969, Theorem 1) for a real symmetric matrix $\mathbf{H}$ and with $\mathbf{H}_{11}^-$ replaced by the Moore-Penrose inverse $\mathbf{H}_{11}^+$. However, his proof remains valid for a complex Hermitian matrix $\mathbf{H}$, and—as pointed out by Bekker (1988, p. 263)—the possibility of using any generalized inverse $\mathbf{H}_{11}^-$ in the product $\mathbf{H}_{12}^*\mathbf{H}_{11}^-\mathbf{H}_{12}$ is ensured by the condition $\mathscr{R}(\mathbf{H}_{12}) \subseteq \mathscr{R}(\mathbf{H}_{11})$; cf. Rao and Mitra (1971, Lemma 2.2.4(iii)).

THEOREM 1. *Let* $\mathbf{A}_1 \in \mathbb{C}_{m,n}$ *and* $\mathbf{B} \in \mathbb{H}_m$. *Then*

$$\mathbf{A}_1\mathbf{A}_1^* \leqslant_{\mathrm{L}} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{0} \leqslant_{\mathrm{L}} \mathbf{B}, \quad \mathscr{R}(\mathbf{A}_1) \subseteq \mathscr{R}(\mathbf{B}), \ \textit{and} \ \lambda_1(\mathbf{A}_1^*\mathbf{B}^-\mathbf{A}_1) \leqslant 1, \quad (2)$$

where $\lambda_1(\cdot)$ *denotes the largest eigenvalue and* $\mathbf{B}^-$ *is any generalized inverse of* $\mathbf{B}$.

Proof. It is clear that

$$\mathbf{0} \leqslant_{\mathrm{L}} \begin{pmatrix} \mathbf{I} & \mathbf{A}_1^* \\ \mathbf{A}_1 & \mathbf{B} \end{pmatrix} \quad \Leftrightarrow \quad \mathbf{0} \leqslant_{\mathrm{L}} \begin{pmatrix} \mathbf{B} & \mathbf{A}_1 \\ \mathbf{A}_1^* & \mathbf{I} \end{pmatrix}, \tag{3}$$

where $\mathbf{I}$ is the identity matrix of an appropriate order. Applying (1) to the two sides of (3) yields (2). ■

Theorem 1 is closely related to Proposition 1 in Baksalary, Liski, and Trenkler (1989). The third condition on the right-hand side of (2) may be reexpressed as $\sigma_1(\mathbf{G}\mathbf{A}_1) \leqslant 1$, where $\sigma_1(\cdot)$ denotes the largest singular value and $\mathbf{G}$ is any matrix such that $\mathbf{G}^*\mathbf{G}$ is a nonnegative definite Hermitian generalized inverse of $\mathbf{B}$ (whose existence is ensured by the nonnegative definiteness of $\mathbf{B}$). Moreover, according to Baksalary and Puntanen (1990, Theorem 1), the second and third conditions in (2) together are equivalent to the requirement that the eigenvalues of $\mathbf{A}_1^*\mathbf{B}^-\mathbf{A}_1$ should be invariant with respect to the choice of $\mathbf{B}^-$ and not exceed one.

Theorem 1 covers some results known in the literature or leads to their generalizations. For instance, substituting $\mathbf{A}_1 = d^{-1/2}\mathbf{a}$, where $\mathbf{a} \in \mathbb{C}_{m,1}$ and d is a positive scalar, we get

$$\mathbf{a}\mathbf{a}^* \leqslant_{\mathrm{L}} d\mathbf{B} \quad \Leftrightarrow \quad \mathbf{0} \leqslant_{\mathrm{L}} \mathbf{B}, \quad \mathbf{a} \in \mathscr{R}(\mathbf{B}), \ \text{and} \ \mathbf{a}^*\mathbf{B}^-\mathbf{a} \leqslant d;$$

cf. Baksalary and Kala (1983, Theorem 1). Substituting $\mathbf{B} = \mathbf{B}_1\mathbf{B}_1^*$ and expressing the condition $\mathscr{R}(\mathbf{A}_1) \subseteq \mathscr{R}(\mathbf{B})$ in the form $\mathbf{A}_1 = \mathbf{B}_1\mathbf{K}$ for some matrix $\mathbf{K}$, the inequality $\lambda_1(\mathbf{A}_1^*\mathbf{B}^-\mathbf{A}_1) \leqslant 1$ transforms to $\lambda_1(\mathbf{K}^*\mathbf{P}_{\mathbf{B}_1^*}\mathbf{K}) \leqslant 1$, where $\mathbf{P}_{\mathbf{B}_1^*}$ is the orthogonal projector on $\mathscr{R}(\mathbf{B}_1^*)$. Consequently, replacing $\mathbf{K}$ in $\mathbf{A}_1 = \mathbf{B}_1\mathbf{K}$ by $\mathbf{K}_0 = \mathbf{P}_{\mathbf{B}_1^*}\mathbf{K}$, it follows that

$$\mathbf{A}_1\mathbf{A}_1^* \leqslant_{\mathrm{L}} \mathbf{B}_1\mathbf{B}_1^* \quad \Leftrightarrow \quad \mathbf{A}_1 = \mathbf{B}_1\mathbf{K}_0 \quad \text{for some contraction } \mathbf{K}_0,$$

i.e., for some $\mathbf{K}_0$ satisfying $\mathbf{K}_0\mathbf{K}_0^* \leqslant_{\mathrm{L}} \mathbf{I}$. The necessity part of this result was proved by Dym (1989, Lemma 0.7) using different arguments.

Extending Lemma 2 of Au-Yeung (1973), Gaffke and Krafft (1982, Theorem 3.5) showed that

$$\mathbf{0} \leqslant_{\mathrm{L}} \mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B} \quad \Rightarrow \quad \mathbf{A}\mathbf{B}^{+}\mathbf{A} \leqslant_{\mathrm{L}} \mathbf{A}. \tag{4}$$

Theorem 1 yields a more general result that if $\mathbf{A} \in \mathbb{H}_m$, $\mathbf{B} \in \mathbb{V}_m$, and $\mathbf{B}^{=}$ is a nonnegative definite Hermitian generalized inverse of $\mathbf{B}$, then

$$\mathbf{A}\mathbf{B}^{=}\mathbf{A} \leqslant_{\mathrm{L}} \mathbf{A} \quad \Leftrightarrow \quad \mathbf{0} \leqslant_{\mathrm{L}} \mathbf{A} \text{ and } \lambda_1(\mathbf{A}\mathbf{B}^{=}) \leqslant 1. \tag{5}$$

In fact, from Theorem 1 it follows that $\mathbf{A}\mathbf{B}^{=}\mathbf{A} \leqslant_{\mathrm{L}} \mathbf{A}$ if and only if $\mathbf{0} \leqslant_{\mathrm{L}} \mathbf{A}$, $\mathscr{R}(\mathbf{A}\mathbf{G}^*) \subseteq \mathscr{R}(\mathbf{A})$, and $\lambda_1(\mathbf{G}\mathbf{A}\mathbf{A}^{-}\mathbf{A}\mathbf{G}^*) \leqslant 1$, where $\mathbf{G}$ is any matrix such that $\mathbf{B}^{=} = \mathbf{G}^*\mathbf{G}$. Observing that $\mathscr{R}(\mathbf{A}\mathbf{G}^*) \subseteq \mathscr{R}(\mathbf{A})$ is fulfilled trivially and $\lambda_1(\mathbf{G}\mathbf{A}\mathbf{G}^*) = \lambda_1(\mathbf{A}\mathbf{G}^*\mathbf{G})$ establishes (5).

Chan and Kwong (1985, p. 539) proposed the conjecture that

$$\mathbf{0} \leqslant_{\mathrm{L}} \mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B} \quad \Rightarrow \quad \mathbf{A}^2 \leqslant_{\mathrm{L}} (\mathbf{A}\mathbf{B}^2\mathbf{A})^{1/2},$$

which was then shown by Furuta (1987, Corollary 2) to follow as a particular case of a more general result established in his Theorem 1; see also Furuta (1989) and Kamei (1988). It turns out that the ordering $\mathbf{A}^2 \leqslant_{\mathrm{L}} (\mathbf{A}\mathbf{B}^2\mathbf{A})^{1/2}$ may hold also when $\mathbf{A}$ and $\mathbf{B}$ are not ordered, an example being

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}.$$

Actually, from Theorem 1 we see that $\mathbf{A}^2 \leqslant_{\mathrm{L}} (\mathbf{A}\mathbf{B}^2\mathbf{A})^{1/2}$ if and only if

$$\mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}\left[(\mathbf{A}\mathbf{B}^2\mathbf{A})^{1/2}\right] \quad \text{and} \quad \lambda_1\left\{\mathbf{A}\left[(\mathbf{A}\mathbf{B}^2\mathbf{A})^{1/2}\right]^{-}\mathbf{A}\right\} \leqslant 1,$$

the former condition being equivalent to the rank equality $\mathrm{r}(\mathbf{A}) = \mathrm{r}(\mathbf{A}\mathbf{B})$.

If $\mathbf{A}, \mathbf{B} \in \mathbb{V}_m$, then Theorem 1 asserts that

$$\mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B} \quad \Leftrightarrow \quad \mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{B}) \text{ and } \lambda_1(\mathbf{A}\mathbf{B}^{+}) \leqslant 1 \tag{6}$$

and

$$\mathbf{A}^2 \leqslant_{\mathrm{L}} \mathbf{B}^2 \quad \Leftrightarrow \quad \mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{B}) \text{ and } \sigma_1(\mathbf{A}\mathbf{B}^{+}) \leqslant 1. \tag{7}$$

Browne (1928) showed that every $\mathbf{K} \in \mathbb{C}_{m,m}$ satisfies $|\lambda_1(\mathbf{K})| \leqslant \sigma_1(\mathbf{K})$; cf. Marshall and Olkin (1979, p. 232). In view of this inequality, comparing the conditions (6) and (7) shows immediately that if $\mathbf{A}, \mathbf{B} \in \mathbb{V}_m$, then

$$\mathbf{A}^2 \leqslant_{\mathrm{L}} \mathbf{B}^2 \quad \Rightarrow \quad \mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B}. \tag{8}$$

The relationship (8) is a particular case of the result due to Löwner (1934), but has also independent simpler proofs; cf., e.g., Davis (1963, p. 199), Gaffke and Krafft (1982, p. 608)—who followed Wigner and Yanase (1964, p. 401), and Marshall and Olkin (1979, p. 464). Our proof, referring to the inequality of Browne (1928), is an addition to this collection.

It is well known that the converse to (8) is in general not true; cf., e.g., Bellman (1960, p. 87), Chan and Kwong (1985, p. 533), and Marshall and Olkin (1979, p. 465). In the particular case when $\mathbf{AB} = \mathbf{BA}$, however, the matrix $\mathbf{AB}^+$ is Hermitian and hence $|\lambda_1(\mathbf{AB}^+)| = \lambda_1(\mathbf{AB}^+) = \sigma_1(\mathbf{AB}^+)$. Then from (6) and (7) it follows that $\mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B}$ together with $\mathbf{AB} = \mathbf{BA}$ implies $\mathbf{A}^2 \leqslant_{\mathrm{L}} \mathbf{B}^2$; cf. Baksalary and Pukelsheim (1991, Theorem 1).

3. MAIN RESULTS

Man (1970, Theorem) showed that if **A**, **B**, and **C** are all positive definite matrices, then the positive definiteness of $\mathbf{BCB} - \mathbf{ACA}$ entails the positive definiteness of $\mathbf{B} - \mathbf{A}$. Moreover, in Corollary 1 he established a converse statement that if $\mathbf{B} - \mathbf{A}$ is positive definite, then $\mathbf{BCB} - \mathbf{ACA}$ is also positive definite for some **C**. We extend these results by allowing **A**, **B**, and **C** to be singular and provide a new proof whose first part is based on the relationship (8).

THEOREM 2. *Let $\mathbf{A}, \mathbf{B} \in \mathbb{V}_m$. Then, for every $\mathbf{C} \in \mathbb{V}_m$ such that $\mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{C})$ and $\mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{C})$,*

$$\mathbf{ACA} \leqslant_{\mathrm{L}} \mathbf{BCB} \quad \Rightarrow \quad \mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B}, \tag{9}$$

and, on the other hand, there exists a nonsingular $\mathbf{C} \in \mathbb{V}_m$ such that

$$\mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B} \quad \Rightarrow \quad \mathbf{ACA} \leqslant_{\mathrm{L}} \mathbf{BCB}.$$

Proof. The left-hand side of (9) is equivalent to $\mathbf{T}^2 \leqslant_{\mathrm{L}} \mathbf{U}^2$, where $\mathbf{T} = \mathbf{C}^{1/2}\mathbf{AC}^{1/2}$ and $\mathbf{U} = \mathbf{C}^{1/2}\mathbf{BC}^{1/2}$. Then (8) yields $\mathbf{T} \leqslant_{\mathrm{L}} \mathbf{U}$, and hence

$\mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B}$, as desired. For the proof of the second part notice that if $\mathbf{Q}$ is a nonnegative definite Hermitian matrix whose range coincides with the null space of $\mathbf{B}$ [e.g., if $\mathbf{Q}$ is the orthogonal projector on the orthocomplement of $\mathscr{R}(\mathbf{B})$], then $\mathbf{C} = \mathbf{B}^{+} + \mathbf{Q}$ is a positive definite matrix for which

$$\mathbf{ACA} = \mathbf{AB}^{+}\mathbf{A} \leqslant_{\mathrm{L}} \mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B} = \mathbf{BCB},$$

where $\mathbf{AQA} = \mathbf{0}$ follows by $\mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{B})$ and $\mathbf{AB}^{+}\mathbf{A} \leqslant_{\mathrm{L}} \mathbf{A}$ follows from (4). ∎

Extending the result of Taylor (1976, Proposition 2), Baksalary, Kala, and Kłaczyński (1983, Theorem 1) showed that, for any $\mathbf{A} \in \mathbb{C}_{m,m}$ and $\mathbf{B} \in \mathbb{V}_{m}$,

$$\mathbf{BA} = \mathbf{BA}^2 \text{ and } \mathbf{A}^*\mathbf{BA} \leqslant_{\mathrm{L}} \mathbf{B} \quad \Leftrightarrow \quad \mathbf{BA} = \mathbf{A}^*\mathbf{BA}. \tag{10}$$

In view of Theorem 1, an alternative version of (10) may be obtained by combining $\mathbf{BA} = \mathbf{BA}^2$ with the two conditions $\mathscr{R}(\mathbf{A}^*\mathbf{B}) \subseteq \mathscr{R}(\mathbf{B})$ and $\lambda_1(\mathbf{AB}^{-}\mathbf{A}^*\mathbf{B}) \leqslant 1$. It turns out, however, that the last condition may be replaced by a weaker inequality, expressing a bound for the trace of $\mathbf{AB}^{-}\mathbf{A}^*\mathbf{B}$ in terms of the rank of this matrix.

THEOREM 3. *For any* $\mathbf{A} \in \mathbb{C}_{m,m}$, $\mathbf{B} \in \mathbb{V}_{m}$, *and any generalized inverse* $\mathbf{B}^{-}$ *of* $\mathbf{B}$, *the following statements are mutually equivalent*:

(a) $\mathbf{BA} = \mathbf{BA}^2$, $\mathscr{R}(\mathbf{A}^*\mathbf{B}) \subseteq \mathscr{R}(\mathbf{B})$, *and* $\lambda_1(\mathbf{AB}^{-}\mathbf{A}^*\mathbf{B}) \leqslant 1$,
(b) $\mathbf{BA} = \mathbf{BA}^2$, $\mathscr{R}(\mathbf{A}^*\mathbf{B}) \subseteq \mathscr{R}(\mathbf{B})$, *and* $\mathrm{tr}(\mathbf{AB}^{-}\mathbf{A}^*\mathbf{B}) \leqslant \mathrm{r}(\mathbf{BA})$,
(c) $\mathbf{BA} = \mathbf{A}^*\mathbf{BA}$.

Proof. Let $\mathbf{B} = \mathbf{B}_1\mathbf{B}_1^*$, and let $\mathbf{S} = \mathbf{B}_1^{+}\mathbf{A}^*\mathbf{B}_1$, where $\mathbf{B}_1^{+}$ is the Moore-Penrose inverse of $\mathbf{B}_1$. The part "(a) $\Rightarrow$ (b)" follows by noting that the last condition in (a) may be expressed as $\lambda_1(\mathbf{S}^*\mathbf{S}) \leqslant 1$, which clearly entails

$$\mathrm{tr}(\mathbf{S}^*\mathbf{S}) \leqslant \mathrm{r}(\mathbf{S}), \tag{11}$$

the last condition in (b). For the proof of the part "(b) $\Rightarrow$ (c)" first notice that if $\mathbf{BA} = \mathbf{BA}^2$ and $\mathscr{R}(\mathbf{A}^*\mathbf{B}) \subseteq \mathscr{R}(\mathbf{B})$, then

$$\mathbf{S}^2 = \mathbf{B}_1^{+}\mathbf{A}^*\mathbf{B}_1\mathbf{B}_1^{+}\mathbf{A}^*\mathbf{B}_1 = \mathbf{B}_1^{+}\mathbf{A}^{*2}\mathbf{B}_1 = \mathbf{S},$$

and hence

$$\mathrm{tr}(\mathbf{S}^2) = \mathrm{r}(\mathbf{S}). \tag{12}$$

Combining (11) with (12) yields $\mathrm{tr}(\mathbf{S}^*\mathbf{S}) \leqslant \mathrm{tr}(\mathbf{S}^2)$, which is impossible unless $\mathbf{S} = \mathbf{S}^*$; cf., e.g., Anderson and Styan (1982, Lemma 3.2). Pre- and postmultiplying $\mathbf{S} = \mathbf{S}^*$ by $\mathbf{B}_1$ and $\mathbf{B}_1^*$, respectively, leads to the equality $\mathbf{A}^*\mathbf{B} = \mathbf{BA}$, and then $\mathbf{BA} = \mathbf{BA}^2$ can be reformulated as $\mathbf{BA} = \mathbf{A}^*\mathbf{BA}$. Finally, if (c) holds, then $\mathbf{A}^*\mathbf{B} = \mathbf{BA}$, and hence $\mathbf{BA} = \mathbf{BA}^2$, $\mathscr{R}(\mathbf{A}^*\mathbf{B}) \subseteq \mathscr{R}(\mathbf{B})$, and $\mathbf{S} = \mathbf{S}^*\mathbf{S}$. Since $\mathbf{S} = \mathbf{S}^*\mathbf{S}$ implies that $\lambda_1(\mathbf{AB}^-\mathbf{A}^*\mathbf{B}) = \lambda_1(\mathbf{S}^*\mathbf{S}) = 1$, the proof is complete. ∎

For any $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$ the matrix $\mathbf{AB}^* + \mathbf{BA}^*$ is Hermitian. With the use of Theorem 1 we obtain the following characterizations of its nonnegative definiteness.

THEOREM 4. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}_{m,n}$. *Then*

$$\mathbf{0} \leqslant_{\mathrm{L}} \mathbf{AB}^* + \mathbf{BA}^* \tag{13}$$

if and only if

$$\lambda_1\left[(\mathbf{A}-\mathbf{B})^*(\mathbf{AA}^* + \mathbf{BB}^*)^-(\mathbf{A}-\mathbf{B})\right] \leqslant 1, \tag{14}$$

or, equivalently,

$$\mathscr{R}(\mathbf{A}:\mathbf{B}) = \mathscr{R}(\mathbf{A}+\mathbf{B}) \quad \textit{and} \quad \lambda_1\left\{\left[(\mathbf{A}:\mathbf{B})^*[(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})^*]^-(\mathbf{A}:\mathbf{B})\right] \leqslant 1, \tag{15}$$

where the choices of the generalized inverses involved are arbitrary.

Proof. The criteria (14) and (15) follow from Theorem 1 as necessary and sufficient conditions for the orderings

$$(\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})^* \leqslant_{\mathrm{L}} (\mathbf{A}:\mathbf{B})(\mathbf{A}:\mathbf{B})^* \leqslant_{\mathrm{L}} (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})^*,$$

which are both equivalent to (13). ∎

In a similar way, by applying Theorem 1 to the orderings

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})^* \leqslant_{\mathrm{L}} (\mathbf{A}:\mathbf{B})(\mathbf{A}:\mathbf{B})^* \leqslant_{\mathrm{L}} (\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})^*,$$

it follows that $\mathbf{AB}^* + \mathbf{BA}^* \leqslant_{\mathrm{L}} \mathbf{0}$ if and only if

$$\lambda_1\left[(\mathbf{A}+\mathbf{B})^*(\mathbf{AA}^*+\mathbf{BB}^*)^-(\mathbf{A}+\mathbf{B})\right] \leqslant 1, \tag{16}$$

or, equivalently,

$$\mathscr{R}(\mathbf{A}:\mathbf{B}) = \mathscr{R}(\mathbf{A}-\mathbf{B}) \quad \text{and} \quad \lambda_1\{\left[(\mathbf{A}:\mathbf{B})^*[(\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})^*]^-(\mathbf{A}:\mathbf{B})\right] \leqslant 1. \tag{17}$$

The range conditions in (15) and (17), which are necessary for the definiteness of $\mathbf{AB}^* + \mathbf{BA}^*$, may be alternatively expressed as the rank conditions $\mathrm{r}(\mathbf{A}:\mathbf{B}) = \mathrm{r}(\mathbf{A}+\mathbf{B})$ and $\mathrm{r}(\mathbf{A}:\mathbf{B}) = \mathrm{r}(\mathbf{A}-\mathbf{B})$, respectively. These conditions have several consequences; cf. Marsaglia and Styan (1974, Corollary 8.1). In particular, they show immediately that if $\mathbf{a},\mathbf{b} \in \mathbb{C}_{m,1}$ are linearly independent, then $\mathbf{ab}^* + \mathbf{ba}^*$ is an indefinite matrix.

In our last result we derive a necessary and sufficient condition for the Löwner ordering between two direct (Kronecker) products of nonnegative definite Hermitian matrices.

THEOREM 5. *Let* $\mathbf{A},\mathbf{B} \in \mathbb{V}_m$ *and* $\mathbf{C},\mathbf{D} \in \mathbb{V}_n$. *Then*

$$\mathbf{A}\otimes\mathbf{C} \leqslant_{\mathrm{L}} \mathbf{B}\otimes\mathbf{D} \tag{18}$$

if and only if the inclusions

$$\mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{B}) \quad \textit{and} \quad \mathscr{R}(\mathbf{C}) \subseteq \mathscr{R}(\mathbf{D}) \tag{19}$$

hold along with the inequality

$$\lambda_1(\mathbf{AB}^-)\lambda_1(\mathbf{CD}^-) \leqslant 1, \tag{20}$$

where $\mathbf{B}^-$ *and* $\mathbf{D}^-$ *are any generalized inverses of* $\mathbf{B}$ *and* $\mathbf{D}$.

Proof. Let $\mathbf{A} = \mathbf{A}_1\mathbf{A}_1^*$ for some $\mathbf{A}_1 \in \mathbb{C}_{m,p}$, and $\mathbf{C} = \mathbf{C}_1\mathbf{C}_1^*$ for some $\mathbf{C}_1 \in \mathbb{C}_{n,q}$. Then (18) may be written in the form

$$(\mathbf{A}_1\otimes\mathbf{C}_1)(\mathbf{A}_1\otimes\mathbf{C}_1)^* \leqslant_{\mathrm{L}} \mathbf{B}\otimes\mathbf{D}. \tag{21}$$

On account of Theorem 1, (21) holds if and only if $\mathbf{0} \leqslant_{\mathrm{L}} \mathbf{B} \otimes \mathbf{D}$,

$$\mathscr{R}(\mathbf{A} \otimes \mathbf{C}) \subseteq \mathscr{R}(\mathbf{B} \otimes \mathbf{D}), \tag{22}$$

and

$$\lambda_1\left[(\mathbf{A}_1 \otimes \mathbf{C}_1)^*(\mathbf{B} \otimes \mathbf{D})^-(\mathbf{A}_1 \otimes \mathbf{C}_1)\right] \leqslant 1. \tag{23}$$

The condition $\mathbf{0} \leqslant_{\mathrm{L}} \mathbf{B} \otimes \mathbf{D}$ is a direct consequence of the assumption that $\mathbf{B} \in \mathbb{V}_m$ and $\mathbf{D} \in \mathbb{V}_n$. The equivalence between (22) and (19) follows from part (c) in Theorem 2.4 of Baksalary, Pukelsheim, and Styan (1989). Since $(\mathbf{B} \otimes \mathbf{D})^-$ may be taken as $\mathbf{B}^- \otimes \mathbf{D}^-$ with any $\mathbf{B}^-$ and $\mathbf{D}^-$, the inequality (23) is equivalent to

$$\lambda_1\left[(\mathbf{A}_1^*\mathbf{B}^-\mathbf{A}_1) \otimes (\mathbf{C}_1^*\mathbf{D}^-\mathbf{C}_1)\right] \leqslant 1. \tag{24}$$

But the eigenvalues of a direct product are the products of the eigenvalues of the two matrices involved. Consequently, since all the eigenvalues of $\mathbf{A}_1^*\mathbf{B}^-\mathbf{A}_1$ and $\mathbf{C}_1^*\mathbf{D}^-\mathbf{C}_1$ are nonnegative, and $\lambda_1(\mathbf{A}_1^*\mathbf{B}^-\mathbf{A}_1) = \lambda_1(\mathbf{A}_1\mathbf{A}_1^*\mathbf{B}^-)$ and $\lambda_1(\mathbf{C}_1^*\mathbf{D}^-\mathbf{C}_1) = \lambda_1(\mathbf{C}_1\mathbf{C}_1^*\mathbf{D}^-)$, the inequality (24) may be reexpressed as in (20). ■

Since $\mathbf{0} \leqslant_{\mathrm{L}} \mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B}$ if and only if $\mathscr{R}(\mathbf{A}) \subseteq \mathscr{R}(\mathbf{B})$ and $\lambda_1(\mathbf{AB}^-) \leqslant 1$, and $\mathbf{0} \leqslant_{\mathrm{L}} \mathbf{C} \leqslant_{\mathrm{L}} \mathbf{D}$ if and only if $\mathscr{R}(\mathbf{C}) \subseteq \mathscr{R}(\mathbf{D})$ and $\lambda_1(\mathbf{CD}^-) \leqslant 1$, it is clear from Theorem 5 that if $\mathbf{0} \leqslant_{\mathrm{L}} \mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B}$ and $\mathbf{0} \leqslant_{\mathrm{L}} \mathbf{C} \leqslant_{\mathrm{L}} \mathbf{D}$, then $\mathbf{A} \otimes \mathbf{C} \leqslant_{\mathrm{L}} \mathbf{B} \otimes \mathbf{D}$. This implication is well known; cf., e.g., Johnson (1978, p. 590) and Marshall and Olkin (1979, p. 467); see also Baksalary, Pukelsheim, and Styan (1989, Theorem 2.5) for a generalization of this result. However, Theorem 5 throws in addition some light on the converse implication, namely, if $\mathbf{A}, \mathbf{B} \in \mathbb{V}_m$ and $\mathbf{C}, \mathbf{D} \in \mathbb{V}_n$ are such that $\mathbf{A} \otimes \mathbf{C} \leqslant_{\mathrm{L}} \mathbf{B} \otimes \mathbf{D}$, then at least one of the relations $\mathbf{A} \leqslant_{\mathrm{L}} \mathbf{B}$ and $\mathbf{C} \leqslant_{\mathrm{L}} \mathbf{D}$ must be satisfied.

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